Accumulation: Thoughts On $f(t) = f(a) + \int_a^t f'(x)\,dx$

By Lin McMullin

The goals of the AP* Calculus program include the statement, "Students should understand the definite integral … as the net accumulation of change…."[1] The Topical Outline includes the topic the "definite integral of the rate of change of a quantity over an interval interpreted as the change of the quantity over the interval:

$$\int_a^b f'(x)\,dx = f(b) - f(a)\text{."}^{2}$$

The equation above is simply the Fundamental Theorem of Calculus, but the topic stresses what it means rather than its use as a technique for evaluating a definite integral. Our equation,

$$f(t) = f(a) + \int_a^t f'(x)\,dx,$$

is the FTC expressed in a slightly different form with a change of variables. It becomes a function defined using an integral. The idea and the use of this equation in this form make a myriad of problems easy to understand and work. Yet the word "accumulation" has, to the best of my recollection, never appeared on the exam in this technical sense, nor does it appear in most popular textbooks.[3] [4]

While there are earlier exams[5] on which this idea can be used, the 2000 exam questions AB2/BC2 (d) and especially AB 4[6] is when this approach made its debut. It can be used on many free-response questions since 2000 and in the multiple-choice questions from the 2003 and 2008 released exams. As with some other topics that appear on the AP calculus exams, since the topic does not appear in the current editions of most of the textbooks used in AP Calculus courses it is

[1] AP Calculus Course Description, The College Board, © 2009, p.7
[2] AP Calculus Course Description, The College Board, © 2009, p.7
[3] This is probably why it is not used on the exams.
[4] "Accumulation" is not mentioned in the indexes of Anton, Finney, Foerster, Hughes-Hallett, Ostebee, Rogawski or Stewart. There is a single sentence in the current edition of Larson, with no exercises using the idea.
[5] For example, 1997 BC 89, see below.
[6] This is a good problem to study. Both the "old" method (solving an initial value differential equation method) and the "new" approach discussed here are shown on the scoring standard. In similar questions in later year only the "new" approach is shown.

important that you use examples from recent exams and make up problems of your own if necessary.

The equation $f(x) = f(a) + \int_a^x f'(t)\,dt$ is, of course, only a simple restatement of the Fundamental Theorem of Calculus. Yet in this form there are a myriad of uses for the equation. On the 2008 AP Calculus exams there are no less than 7 questions where this can be used, yet, as mentioned above, this form and this approach is not mentioned in the most textbooks. It appears 10 times on the 2010 AB scoring standards.[7] We will consider some of the uses of this equation, with some examples from the 2008 and 2009 AB Calculus exams shortly.

The equation says that the function value is equal to some starting value (which may be zero) plus the accumulated change.

Final Value = Starting Value + Accumulated Change

$$f(t) = f(a) + \int_a^t f'(x)\,dx$$

The integral $\int_a^t f'(x)\,dx$ gives the accumulated change (or the net change) from some starting point at *a* to *t* of a function, *f*, in terms of its derivative, f'. When we add the starting value, $f(a)$ to the integral we have the value of the function at *t*.

If the derivative is the velocity, $v(t)$, of a moving object then the integral gives the *displacement* over the time interval [*a*, *t*]. Here the equation gives the position, $s(t)$, of the moving object at time *t*. The object starts at some position $s(a)$. The equation then looks like this:

$$s(t) = s(a) + \int_a^t v(x)\,dx$$

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[7] 2010 A 1(c) is a bit of overkill; the question is at best an Algebra 2 question. However, it is a good, easy, example of the idea.

The first time you saw this equation was for a functions whose rate of change (derivative) was a constant, usually called the slope, *m*. If you know one point of the function, say (x_0, y_0), then the *y*-value anywhere on the function is

$$y(x) = y_0 + \int_{x_0}^{x} m\,dt = y_0 + mt\Big|_{x_0}^{x} = y_0 + mx - mx_0$$
$$y = y_0 + m(x - x_0)$$

This is, of course, the *point-slope equation* of a line. You certainly didn't use the integral in Algebra 1, but the point-slope form is just a special case of our equation. The final value, *y*, is equal to the starting value, y_0, plus the accumulated change. The accumulated change, the change in the *x*-values, is the rate of change, $m = \dfrac{\text{change in } y}{\text{change in } x}$ multiplied by, the actual change in *x*, $x - x_0$.

In calculus we deal with variable slopes, variable rates of change. Here are two similar examples from the 2008 AB Exam. In both the derivative is called the velocity of a moving object.

- <u>2008 AB 7</u>: A particle moves along the *x*-axis with velocity given by $v(t) = 3t^2 + 6t$ for time $t \geq 0$. If the particle is at position *x* = 2 at time *t* = 0, what is the position of the particle at time $t = 1$? (Multiple-choice answers omitted.)
  Using our equation, the solution can be written immediately:

$$x(1) = 2 + \int_0^1 3t^2 + 6t\,dt = 2 + \left(t^3 + 3t^2\right)\Big|_0^1 = 2 + 4 = 6$$

- <u>2008 AB 87</u>: An object traveling in a straight line has position $x(t)$ at time *t*. If the initial position is $x(0) = 2$ and the velocity of the object is $v(t) = \sqrt[3]{1 + t^2}$, what is the position of the object at time *t* = 3? (Multiple-choice answers omitted.)
  The wording is almost identical to the previous example as is the solution. The integration is done by calculator.

$$x(3) = 2 + \int_0^3 \sqrt[3]{1 + t^2}\,dt \approx 2 + 4.51153 \approx 6.512$$

These two could have been approached as initial value problems: $\frac{dy}{dx} = f'(x)$ with the initial condition $(a, f(a))$. The "old" approach is to find an antiderivative including a constant *C*, use the initial condition to evaluate *C*, write the particular solution, and finally evaluate the solution at $x = a$. The "new" approach, which can be used with any first-order differential equation where the derivative is a function of only one variable, is to use the equation.

$$y(x) = f(a) + \int_a^x \left(\tfrac{dy}{dt}\right) dt = f(a) + \int_a^x f'(t)\,dt$$

Here's a similar question from a long time ago.

- 1997 BC 89 If f is an antiderivative of $\dfrac{x^2}{1+x^5}$ such that $f(1) = 0$, then $f(4) =$

  (Multiple-choice answers omitted.) Once again the integration is done by calculator especially since the antiderivative is very difficult to compute.

$$f(4) = f(1) + \int_1^4 \frac{x^2}{1+x^5}\,dx \approx 0.376$$

Here is another example from the 2008 AB exam that forces students to recognize this form with a generic function

- 2008 AB 81: If $G(x)$ is an antiderivative for $f(x)$ and $G(2) = -7$, then $G(4) =$

  (A) $f'(4)$ (B) $-7 + f'(4)$ (C) $\int_2^4 f(t)\,dt$

  (D) $\int_2^4 \left(-7 + f(t)\right) dt$ (E) $-7 + \int_2^4 f(t)\,dt$

  Answer (E)

On the 2008 AP Calculus tests the equation could be used on all of these questions:

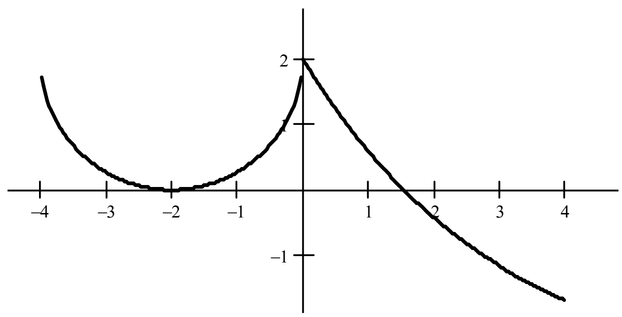
- AB Multiple-choice 7, 81 and 87
- Free-response: AB 2 / BC 2 (d), AB 3 (c), AB 4 / BC 4 (a) *twice* and BC 5 (c).

While other approaches may be possible, any time you are given a starting value and a derivative (rate of change, slope, etc.) this method will get you the answer quickly. Free-response questions often can be solved using our equation. Here is a typical problem, one of many.

2009 AB 6 gave the derivative of a function *f* as the piecewise defined function

$$f'(x)=\begin{cases} g(x) & \text{for } -4\le x\le 0 \\ 5e^{-x/3}-3 & \text{for } 0<x\le 4 \end{cases}$$

where $g(x)$ was the semicircle shown on the graph below. The *x*-intercepts of f' are $x = -2$ and $x=3\ln\left(\frac{5}{3}\right)$. The initial condition is $f(0)=5$. (All given)



In part (b) students were asked to find $f(4)$ and $f(-4)$. Using our approach the results are straight forward:

$$f(4)=5+\int_0^4 5e^{-x/3}-3\,dx=5+\left(-15e^{-x/3}-3x\right)\Big|_0^4=8-15e^{-4/3}$$

$$f(-4)=f(0)+\int_0^{-4} g(x)\,dx=f(0)-\int_{-4}^{0} g(x)\,dx=5-(8-2\pi)=2\pi-3$$

The second definite integral is most easily found by subtracting the area of the semicircle, 2π, from the area of a rectangle, 8, drawn around it. Notice that $\int_0^{-4} g(x)\,dx$ is “backwards”: the upper limit of integration is less than the lower limit. Therefore, its value is the opposite of the area of the region between the semicircle and the *x*-axis.

Integration questions with the lower limit of integration greater than the upper limit, and where the values must be found from a graph are confusing for students. A way of looking at this situation is to reason this way: We are looking for $f(-4)$. If we integrate starting at $x = -4$ we have accumulated the amount $(8-2\pi)$ by the time we get to $x = 0$ where the value is 5 (given). So if we *subtract* the $(8-2\pi)$ from 5, we will have our starting value $f(-4)$. Symbolically, the details look like this and this avoids the integral with the lower limit greater than the upper.

$$\begin{aligned} f(-4)+\int_{-4}^{0} g(x)\,dx &= f(0) \\ f(-4)+(8-2\pi) &= 5 \\ f(-4) &= 5-(8-2\pi) \end{aligned}$$

Part (c) of this same question asked students to find the x value at which f has its absolute maximum. The maximum occurs at $x = 3\ln\left(\frac{5}{3}\right)$ since this is the only place where the derivative of f changes from positive to negative. While the preceding sentence receives full credit, here is another approach using the Candidate's Test and our equation:

> The candidates for the location of the maximum are $x = -4$, $x = 3\ln\left(\frac{5}{3}\right) = M$, and $x = 4$.
>
> $f(M) = f(-4)+\int_{-4}^{M} f'(x)\,dx.$ Since $f'(x) \ge 0$ on the interval $[-4, M]$ the integral is positive, it follows that $f(M) > f(-4)$
>
> $f(4) = f(M)+\int_{M}^{4} 5e^{-x/3}-3\,dx$. Since on the interval $[M, 4]$ the integral is negative, it follows that $f(M) > f(4)$
>
> Therefore, the maximum occurs at $x = M$.

Admittedly, this is overkill, but it shows another use of the concept. And a few students did use this approach on the exam!

If you look through the AP Calculus exams, especially from 2000 on, you will find many examples where this equation can be used to advantage; if you look through the textbooks you will find no such examples. While there are always other ways to approach these problems, this one is straightforward, easy to understand and applicable in a variety of situations. You need to supplement your textbook so that students will understand and be able to use this approach. But whether you are teaching AP or not this approach make so many problems so much easier to understand and compute.

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